Math 409 Midterm 1

Name: ____________________

This exam has 4 questions, for a total of 100 points + 8 bonus points.

Please answer each question in the space provided. No aids are permitted.

Question 1. (40 pts)

In each of the following eight cases, indicate whether the given statement is true or false. No justification is necessary.

(a) The set $\left\{\frac{n}{2^m} : \forall n, m \in \mathbb{N}\right\}$ is countable.

Solution: True.

(b) Let F be a nonempty subset of $\mathbb{R}$. If there exists an injective map $f\colon [0,1] \to F$, then F is uncountable.

Solution: True.

(c) Let E be a nonempty subset of $\mathbb{R}$ with an upper bound, then the set $\{x^2 : x \in E\}$ also has a upper bound.

Solution: False. Here is a counterexample: $E = \{\text{all negative integers}\}$. E is bounded above by 0, but $\{x^2 : x \in E\}$ is does not have an upper bound.

(d) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \sin(x)$. Suppose $E = [1, \infty)$, then $f^{-1}(E) = \{2k\pi : k \in \mathbb{Z}\}$.

Solution: False. $f^{-1}(E) = \{2k\pi + \pi/2 : k \in \mathbb{Z}\}$.

(e) For a given function $f\colon \mathbb{R} \to \mathbb{R}$ and a subset A of $\mathbb{R}$, we have $f^{-1}(f(A)) = A$.

Solution: False. In general, we only have $A \subseteq f^{-1}(f(A))$.

(f) Let $a, b \in \mathbb{R}$. Then $a \leq b$ if and only if $a < b + \frac{1}{n}$ for all $n \in \mathbb{N}$.

Solution: True.

(g) The function $f(x) = x^2 + 2x - 5$ is injective on the set $(-\infty, -6]$. That is, as a function $f\colon (-\infty, -6] \to \mathbb{R}$, it is injective.

Solution: True. This is because $f'(x) = 2x+2 < 0$ for all $x \in (-\infty, -6]$, which implies f is decreasing on $(-\infty, -6]$.

(h) There is a bijection from $\mathbb{Q}$ to $\mathbb{N}$.

Solution: True.

Question 2. (25 pts)

(a) State the well-ordering principle.

Solution: If E is a nonempty subset of $\mathbb{N}$, then E has a least element.

(b) Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Solution: When $n = 1$, LHS $= 1 =$ RHS.
Suppose we have proved the equality for n. Consider the case of $n + 1$:

$$\begin{aligned} \text{LHS} &= 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{(n+1)(n+2)(2n+3)}{6} = \text{RHS} \end{aligned}$$

This finishes the proof.

Question 3. (35 pts)

(a) State the completeness axiom for $\mathbb{R}$.

Solution: For every nonempty subset $E \subset \mathbb{R}$, if E is bounded above, then E has a finite supremum.

(b) Let E be a nonempty bounded subset of $\mathbb{R}$. Define A to be the set $\{x^4 : x \in E\}$. Prove that $\sup A$ exists and express $\sup A$ in terms of $\sup E$ and $\inf E$.

Solution: Since E is bounded, then there exists $M > 0$ such that $|x| \leq M$ for all $x \in E$. This implies that $x^4 \leq M^4$. So $A = \{x^4 : x \in E\}$ is bounded above. By the completeness axiom, $\sup A$ exists.

$$\sup A = \max\{(\inf E)^4, (\sup E)^4\}.$$

(c) Let B be a nonempty subset of $\mathbb{R}$. Suppose every element of B is positive. Define D to be the set $\{b^{-1} : b \in B\}$. Prove that $\inf D > 0$ if and only if B is bounded.

Solution: $\inf D > 0$
$\Longleftrightarrow$ there exists $\varepsilon > 0$ such that $b^{-1} > \varepsilon$ for all $b \in B$
$\Longleftrightarrow$ $\varepsilon^{-1} > b > 0$ for all $b \in B$
$\Longleftrightarrow$ B is bounded.

Bonus Question 4. (8 pts)

Suppose P is the set of all polynomials with rational coefficients. That is,

$$P = \left\{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \ : \ n \in \mathbb{N} \text{ and } a_i \in \mathbb{Q}\right\}.$$

Prove that P is a countable set.

Solution: Let $A_k = \underbrace{\mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}}_{k \text{ copies}}$. Then the union $E = \bigcup_{k \geq 1} A_k$ is countable. There is a injection from P to E by

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mapsto (a_0, a_1, \cdots, a_n) \in A_{n+1} \subset E.$$

This implies that P is countable.